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The bifurcation theory of magnetic monopoles in a charged two-condensate Bose–Einstein system

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Abstract

Magnetic monopoles, which are particle-like field configurations with which one can associate a topological charge, widely exist in various three-dimensional condensate systems. In this paper, by making use of *Duan*'s topological current theory, we obtain the charge density of magnetic monopoles and their bifurcation theory in a charged two-condensate Bose–Einstein system. The evolution of magnetic monopoles is studied from the topological properties of a three-dimensional vector field. The magnetic monopoles are found generating or annihilating at the limit points and encountering, splitting or merging at the bifurcation points.

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1. Introduction

An elementary particle with a net magnetic charge is an old hypothetical particle called magnetic monopole which arises in classical electromagnetism and has never been seen in the real world. Modern interest in the magnetic monopole focuses on quantum field theory, notably grand unified theories and superstring theories, that predict the existence of the possibility of magnetic monopoles. In 1931, Dirac [1] proposed that the magnetic monopole with an attached Dirac string may exist in quantum electrodynamics by their phenomenon of electric charge quantization. In 1974, it was shown by 't Hooft [2] and Polyakov [3] that a magnetic monopole could be regarded as topological excitations in a quantum field theory due to the spontaneous symmetry breaking mechanism. The quantized magnetic charge was interpreted as the topological charge of the magnetic monopole. After 't Hooft and Polyakov's works, Duan and Ge [4] studied the rigorous topological expressions of many moving magnetic monopoles, which could not be derived from 't Hooft and Polyakov's theory. It also revealed the inner structures of the magnetic charge density current and showed that the zero points of the Higgs field were a point-like source of the magnetic monopole. Recently, the theory of the magnetic monopole has been frequently employed in studying the grand unified theories, the phase transitions in the early universe, and the topological excitations in condensed matter physics.

In condensed matter physics, there are also topological objects that imitate magnetic monopoles. In chiral superconductors and superfluids, the magnetic monopole excitations have been well studied by Volovik [5], and such a magnetic monopole is the analog of a Dirac magnetic monopole which combined with two Abrikosov vortices or four half-quantum vortices. These vortex lines represent the ‘conventional’ Dirac string. Such Dirac-like monopoles has been investigated also in ferromagnetic spinor Bose–Einstein condensates [6]. Besides the analog of the Dirac magnetic monopole, the ‘ ’t Hooft–Polyakov monopole’ can also be introduced to condensed matter physics [8, 9]. In spinor Bose–Einstein antiferromagnets, such a point-like monopole has recently been worked out by a number of authors [7, 8]. Moreover, in a charged two-condensate Bose–Einstein system, such a monopole which has a quantized magnetic charge and can be regarded as a real magnetic monopole has been proposed recently by Jiang [9]. The induced magnetic field of a magnetic monopole and their rigorous density distribution expression have been deduced by using *Duan*’s topological current theory [19, 20]. As indicated in the above paragraph, the magnetic monopole excitations have already been studied in the context of quantum field theory. Therefore, as pointed out in [7], magnetic monopole excitations in condensed matter offer the exciting opportunity to study the properties of magnetic monopoles in detail. Undoubtedly, this will lead to important new insights into the general topic of topological excitations in a quantum field theory.

Furthermore, two-gap superconductivity has drawn great interest recently due to the discovery of the two-band superconductor with surprisingly high critical temperature MgB2 [10]. Two-gap superconductivity is being supported by an increasing number of experimental reports. Principally, the two-gap superconductivity can be investigated in the frame of a charged two-condensate Bose system [9, 11, 12]. This system is described by a Ginzburg–Landau model with two flavors of Cooper pairs. Alternatively, it relates to a Gross–Pitaevskii functional with two charged condensates of tightly bound fermion pairs, or some other charged bosonic fields. Such theoretical models have a wider range of applications, including interference between two Bose condensates [13], a multiband superconductor [14], two-component Bose–Einstein condensates [15] and a superconducting gap structure of spin-triplet superconductor Sr2RuO4 [16]. Using this theoretical model, two typical topological excitations have been presented. One is the knotted vortices [11, 12], and the other is the magnetic monopoles [9]. The main purpose of this paper is to discuss the topological properties of the magnetic monopole excitations in a charged two-condensate Bose system.

In [9], Jiang has proposed the magnetic monopole excitation in a charged two-condensate Bose system, and by using the *Duan*’s topological current theory, the rigorous density distribution expression of the magnetic monopole has been deduced. The topological charges of magnetic monopoles can be expressed in terms of the Hopf indices and Brouwer degrees. However, Jiang’s conclusions are based on a very important condition that the Jacobian $D(\phi/x) \neq 0$ must be satisfied. When this condition fails, what will happen? In this paper, we will investigate the behavior of the magnetic monopole when this condition fails.

This paper is arranged as follows. In section 2, we give a prime view of the derivation of the topological structure of magnetic monopoles. The magnetic monopoles are quantized at the topological level and their quantum numbers are determined by the Hopf indices and Brouwer degree. In section 3, we introduce the generation and annihilation of magnetic monopoles at the limit point. The bifurcation theory of magnetic monopoles at the first- and second-order degenerate points are investigated in sections 4 and 5, respectively. Section 6 gives our conclusions.

2. Magnetic monopole excitations in a charged two-condensate Bose–Einstein system: a prime introduction

In order to make the background of this paper clear, in this section we will give a brief review of the magnetic monopole excitations in a charged two-condensate Bose–Einstein system. First, let us consider a Bose–Einstein system with two electromagnetically coupled, oppositely charged condensates, which can be described by a two-flavor (denoted by $\alpha = 1, 2$) Ginzburg–Landau or Gross–Pitaevskii (GLGP) functional [12], whose free energy density is given by

$$F = \frac{1}{2m_1}\left|\left(\hbar\partial_\mu + \mathrm{i}\frac{2e}{c}A_\mu\right)\Psi_1\right|^2 + \frac{1}{2m_2}\left|\left(\hbar\partial_\mu - \mathrm{i}\frac{2e}{c}A_\mu\right)\Psi_2\right|^2 + V(\Psi_{1,2}) + \frac{\vec{B}^2}{8\pi}, \tag{1}$$

in which

$$V(\Psi_{1,2}) = -b_\alpha|\Psi_\alpha|^2 + \frac{c_\alpha}{2}|\Psi_\alpha|^4 + \eta[\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1], \tag{2}$$

where η is a characteristic of interband Josephson coupling strength [17]. The properties of the corresponding model with a single charged two-condensate Bose–Einstein system are well known. And the relevant field degrees of freedom are the massive coefficient of the single complex order parameter and a vector field that gains a mass because of the Meissner–Higgs effect. What is very important in the present GLGP model is that the two charged fields are not independent but nontrivially coupled through the electromagnetic field, which indicate that there should be a nontrivial, hidden topology in this system. However, it cannot be recognized obviously in the form of equation (1). For working out the topological structure and studying it conveniently, we need to reform the GLGP functional. Babaev *et al* [12] introduce a set of variables ρ and $\chi_{1,2}$ by

$$\Psi_\alpha = \sqrt{2m_\alpha}\rho\chi_\alpha, \tag{3}$$

where the complex $\chi_\alpha = |\chi_\alpha|\mathrm{e}^{\mathrm{i}\varphi_\alpha}$ satisfying $|\chi_1|^2 + |\chi_2|^2 = 1$ and ρ has the following expression

$$\rho^2 = \frac{1}{2}\left(\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right), \tag{4}$$

where ρ is a massive field which is related to the densities of the Cooper pair. Using the variables $\chi_{1,2}$ and Pauli matrices σ, we define the three-dimensional unit vector field $\vec{n} = (\bar{\chi}, \vec{\sigma}\chi)$, where (,) denotes the scalar product and $\bar{\chi} = (\chi_1^*\chi_2^*)$. Then the original GLGP free energy density equation (1) can be represented as

$$\begin{aligned} F = {} & \frac{\hbar^2\rho^2}{4}(\partial\vec{n})^2 + \hbar^2(\partial\rho)^2 + \frac{\rho^2}{16}\vec{C}^2 + V(\rho, n_1, n_3) \\ & + \frac{\hbar^2c^2}{512\pi e^2}\left(\frac{1}{\hbar}[\partial_\mu C_\nu - \partial_\nu C_\mu] - \vec{n}\cdot\partial_\mu\vec{n}\times\partial_\nu\vec{n}\right)^2, \end{aligned} \tag{5}$$

where

$$C_\mu = 2\mathrm{i}\hbar[\chi_1\partial_\mu\chi_1^* - \chi_1^*\partial_\mu\chi_1 - \chi_2\partial_\mu\chi_2^* + \chi_2^*\partial_\mu\chi_2] - \frac{8e}{c}A_\mu. \tag{6}$$

Now we find that there exists an exact equivalence between the two-flavor GLGP model and the nonlinear $O(3)\sigma$ model [18] that is much more important to describe the topological structure in high energy physics. In this paper, based on *Duan-Ge*'s decomposable gauge potential theory and *Duan*'s topological current theory, we display that there exists another kind of topological defect, namely the magnetic monopoles in this system.

As shown in equation (5), we know that the magnetic field of the system can be divided into two parts. One part, is the contribution of field C_μ which is introduced by the supercurrent density and can only present us with the topological defects named vortices, as what is in the single-condensate system. The other part is the contribution $\vec{n}\cdot\partial_\mu\vec{n}\times\partial_\nu\vec{n}$ to the magnetic field, which originates from interactions of Cooper pairs of two different flavors and is a fundamentally important property of the two-condensate system.

The induced magnetic field B_μ due to $\vec{n}\cdot\partial_\mu\vec{n}\times\partial_\nu\vec{n}$ term is expressed as

$$B^\mu = \frac{\hbar c}{8\pi e}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}n^a\partial_\nu n^b\partial_\lambda n^c. \tag{7}$$

Then, the divergence of the induced magnetic field, namely Q, can be represented in terms of the unit vector field n^a as

$$Q = \partial_\mu B^\mu = \frac{\hbar c}{8\pi e}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}\partial_\mu n^a\partial_\nu n^b\partial_\lambda n^c, \tag{8}$$

this is just the magnetic charge density of the system ρ_m, which is the time component of the topological current

$$J_m^\mu = \frac{\hbar c}{8\pi e}\epsilon^{\mu\nu\lambda\rho}\epsilon_{abc}\partial_\nu n^a\partial_\lambda n^b\partial_\rho n^c, \qquad (\mu,\nu,\lambda,\rho = 0,1,2,3). \tag{9}$$

It is easy to see that the current (9) is identically conserved,

$$\partial_\mu J_m^\mu = 0. \tag{10}$$

In order to investigate the topological structure of the magnetic charge current, we introduce a three-component vector order parameter $\vec{\phi} = (\phi^1,\phi^2,\phi^3)$ formed by the unit vector $\vec{n}$ which satisfies

$$n^a = \frac{\phi^a}{\|\phi\|}, \qquad \|\phi\| = \sqrt{\phi^a\phi^a}, \qquad (a = 1,2,3). \tag{11}$$

Obviously, the order parameter $\vec{\phi}$ can be looked upon as a smooth mapping between the three-dimensional space X (with the local coordinate x) and the three-dimensional Euclidean space $R^3\phi : x \longmapsto \vec{\phi}(x) \in R^3$. n^a is a section of sphere bundle $S(X)$.

Applying *Duan*'s topological current theory [19, 20], one can obtain

$$J_m^\mu = \frac{\hbar c}{e}\delta^3(\vec{\phi})D^\mu\left(\frac{\phi}{x}\right), \tag{12}$$

and the Jacobian $D^\mu\left(\frac{\phi}{x}\right)$ is defined as

$$\epsilon^{abc}D^\mu\left(\frac{\phi}{x}\right) = \epsilon^{\mu\nu\lambda\rho}\partial_\nu\phi^a\partial_\lambda\phi^b\partial_\rho\phi^c. \tag{13}$$

The delta function expression (12) of the topological current J_m^μ tells us that it does not vanish only at the zero points of $\vec{\phi}$, i.e., the sites of the magnetic monopole. The implicit function theorem [21] shows that under the regular condition

$$D^0\left(\frac{\phi}{x}\right) \neq 0, \tag{14}$$

the general solutions of

$$\phi^a(x^1,x^2,x^3,t) = 0, \qquad (a = 1,2,3). \tag{15}$$

The solutions of equation (15) can be generally expressed as

$$x^1 = x_i^1(t), x^2 = x_i^2(t), x^3 = x_i^3(t), \qquad (i = 1,2,\ldots,K)$$

that represent the world lines of K isolated zero points $\vec{z}_i(t)(i = 1, 2, \ldots, K)$. These zero points are just the magnetic monopole excitations, and the ith world line $\vec{z}_i(t)$ determines the motion of the ith magnetic monopole.

The δ-function theory [22] demonstrates the relation

$$\delta^3(\vec{\phi}) = \sum_{i=1}^{K} \frac{\beta_i}{\left|D\left(\frac{\phi}{x}\right)\right|_{\vec{z}_i}} \delta^3(\vec{r} - z_i(t)),$$

where the positive integer β_i is the Hopf index of ϕ-mapping, which means that when $\vec{r}$ covers the neighborhood of the zero point $\vec{z}_i(t)$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ space β_i times, which is a topological number of first Chern class and relates to the generalized winding number of the ϕ-mapping. With the definition of the vector Jacobian (13), and using the implicit function theorem, the general velocity of the ith magnetic monopole can be introduced

$$V_i^{\mu} = \frac{\mathrm{d}z_i^{\mu}}{\mathrm{d}t} = \left.\frac{D^{\mu}\left(\frac{\phi}{x}\right)}{D\left(\frac{\phi}{x}\right)}\right|_{\vec{z}_i}, \qquad V_i^0 = 1. \tag{16}$$

Then, we can get the magnetic charge current J_m^{μ} in the form of the current and the density of a system of K classical point particles in (3+1)-dimensional spacetime with topological charge $W_i = \beta_i \eta_i$

$$\begin{aligned} \vec{j}_m &= \frac{\hbar c}{e} \sum_{i=1}^{K} W_i \vec{V}_i \delta^3(\vec{r} - \vec{z}_i(t)), \\ \rho_m &= \frac{\hbar c}{e} \delta^3(\vec{\phi}) D\left(\frac{\phi}{x}\right) = \frac{\hbar c}{e} \sum_{i=1}^{K} W_i \delta^3(\vec{r} - \vec{z}_i(t)), \end{aligned} \tag{17}$$

where $\eta_i = sgn\big(D\big(\frac{\phi}{x}\big)|_{\vec{z}_i}\big) = \pm 1$ is the Brouwer degree, and $W_i = \beta_i \eta_i$ is the winding number of $\vec{\phi}$ at the zero point $\vec{z}_i(t)$. It is clear that equation (17) describes the motion of the magnetic monopoles in spacetime, and the topological quantum numbers are determined by the Hopf indices β_i and Brouwer degrees η_i of the ϕ-mapping at its zeros. Here, $\eta_i = +1$ corresponds to a magnetic monopole and $\eta_i = -1$ corresponds to an anti-magnetic monopole.

3. The generation and annihilation of magnetic monopoles

As investigated before, the equations of $\vec{\phi}$'s zeros play an important role in describing the topological structures of the magnetic monopole in a charged two-condensate Bose–Einstein system. Now we begin discussing the properties of the zero points, in other words, the properties of the solutions of equation (15). As we knew before, if the Jacobian

$$D^0\left(\frac{\phi}{x}\right) \neq 0, \tag{18}$$

we will have the isolated zeros of the vector field $\vec{\phi}$. The isolated solutions are called regular points. However, when the condition (18) fails, the usual implicit function theorem [21] is of no use. The above discussion will change in some way and lead to the branch process. Now, we denote one of the zero points as $(t^*, \vec{x}^*)$. Let us explore what happen to the magnetic monopoles. In *Duan*'s topological current theory, there are usually two kinds of branch points, the limit points and bifurcation points, satisfying

$$D^i\left(\frac{\phi}{x}\right)\bigg|_{(t^*, \vec{x}^*)} \neq 0, \qquad i = 1, 2, 3 \tag{19}$$

and

$$D^i\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x^*})} = 0, \qquad i = 1, 2, 3, \tag{20}$$

respectively. Here, we consider the case (19). The other case (20) is complicated and will be treated in sections 3 and 4. In order to be simple and without losing generality, we choose $i = 1$.

If the Jacobian

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x^*})} \neq 0, \tag{21}$$

we can use the Jacobian $D^1\left(\frac{\phi}{x}\right)$ instead of $D^0\left(\frac{\phi}{x}\right)$ for the purpose of using the implicit function theorem. This means we will replace the timelike variable $x^0 = t$ by x^1. For seeing this point clearly, we rewrite the equations of (15) as

$$\vec{\phi}(x^1, x^2, x^3, t) = 0. \tag{22}$$

Then we have a unique solution of equation (15) in the neighborhood of the limit point $(t^*, \vec{x^*})$

$$t = t(x^1), \qquad x^2 = x^2(x^1), \qquad x^3 = x^3(x^1), \tag{23}$$

with $t^* = t(x^{1*})$. We call the critical points $(t^*, \vec{x^*})$ the limit points. In the present case, we know that

$$\frac{\mathrm{d}x^1}{\mathrm{d}t}\bigg|_{(t^*,\vec{x^*})} = \frac{D^1\left(\frac{\phi}{x}\right)}{D\left(\frac{\phi}{x}\right)}\bigg|_{(t^*,\vec{x^*})} = \infty, \tag{24}$$

i.e.,

$$\frac{\mathrm{d}t}{\mathrm{d}x^1}\bigg|_{(t^*,\vec{x^*})} = 0. \tag{25}$$

Then the Taylor expansion of $t = t(x^1)$ at the limit point $(t^*, \vec{x^*})$ is

$$t - t^* = \frac{1}{2}\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}\bigg|_{(t^*,\vec{x^*})}\left(x^1 - z_l^1\right)^2, \tag{26}$$

which is a parabola in the $x^1 - t$ plane. From equation (26) we can obtain two solutions $x_1^1(t)$ and $x_2^1(t)$, which give two branch solutions (world lines of magnetic monopoles). If

$$\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}\bigg|_{(t^*,\vec{x^*})} > 0. \tag{27}$$

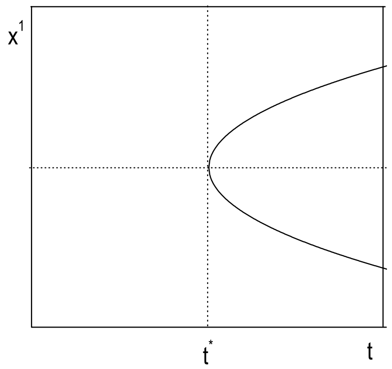
We have the branch solutions for $t > t^*$ (see figure 1(*a*)); otherwise, we have the branch solutions for $t < t^*$ (see figure 1(*b*)). These two cases are related to the origin and annihilation of magnetic monopoles.

One of the results of equation (24), that the velocity are infinite when they are annihilating, agrees with the fact obtained by Bray [23] who has a scaling argument associated with the point defects final annihilation which leases to a large velocity tail. From equation (24), we also obtain a new result that the velocity field is infinite when they are generating, which is gained only from the topology of the vector function $\vec{\phi}$.

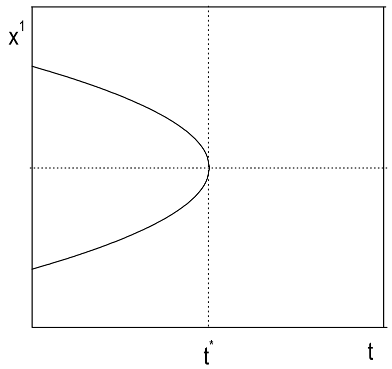
Since topological current is identically conserved, the topological charges of these two generated or annihilated magnetic monopoles must be opposite at the limit point, i.e.,

$$\beta_{l_1}\eta_{l_1} = -\beta_{l_2}\eta_{l_2}, \tag{28}$$

which shows that $\beta_{l_1} = \beta_{l_2}$ and $\eta_{l_1} = -\eta_{l_2}$, which is important in the charged two-component Bose–Einstein system. One can see the fact that the Brouwer degree η is indefinite at the



(*a*)



(*b*)

Figure 1. Projecting the world lines of magnetic monopoles onto the $(x^1 - t)$ plane. (*a*) The branch solutions for equation (26) when $\mathrm{d}^2t/(\mathrm{d}x^1)^2|_{(t^*,\vec{z}_l)} > 0$, i.e., two magnetic monopoles with opposite charges generate at the limit point, i.e., the origin of magnetic monopoles. (*b*) The branch solutions for equation (26) when $\mathrm{d}^2t/(\mathrm{d}x^1)^2|_{(t^*,\vec{z}_l)} < 0$, i.e., two magnetic monopoles with opposite charges annihilate at the limit point.

limit points implies and can change discontinuously at limit points along the world lines of the magnetic monopoles (from ± 1 to ∓ 1).

For a limit point it is required that $D^1\left(\frac{\phi}{x}\right)\big|_{(t^*,\vec{x}^*)} \neq 0$. As to a bifurcation point [24], it must satisfy a more complex condition. This case will be discussed in the following section.

4. Bifurcation of magnetic monopoles

In this section we have the restrictions of equation (20) at the bifurcation points $(t^*, \vec{x}^*)$,

$$D\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = 0, \qquad D^i\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = 0, \qquad i = 1, 2, 3, \tag{29}$$

which leads to an important fact that the function relationship between t and $\vec{x}$ is not unique in the neighborhood of the bifurcation point $(t^*, \vec{x}^*)$. In our dynamic form of charge current, this fact can be seen easily from equation (16)

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{D^i\left(\frac{\phi}{x}\right)}{D\left(\frac{\phi}{x}\right)}\Bigg|_{(t^*,\vec{x}^*)}, \qquad i = 1, 2, 3, \tag{30}$$

which under equation (29) directly shows that the direction of the integral curve of equation (30) is indefinite at $(t^*, \vec{x^*})$, i.e., the velocity field of the magnetic monopoles is indefinite at $(t^*, \vec{x^*})$. That is why the very point $(t^*, \vec{x^*})$ is called a bifurcation point.

Assume that the bifurcation point $(t^*, \vec{x^*})$ has been found from equation (15) and (29). We know that, at the bifurcation point, the rank of the Jacobian matrix $\left[\frac{\partial\phi}{\partial x}\right]$ is less than 3. We suppose

$$\operatorname{rank}\left[\frac{\partial\phi}{\partial x}\right]\bigg|_{(t^*,\vec{x^*})} = 3 - 1 = 2, \tag{31}$$

and let

$$D^i\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x^*})} = \begin{vmatrix} \dfrac{\partial\phi^1}{\partial x^2} & \dfrac{\partial\phi^1}{\partial x^3} \\ \dfrac{\partial\phi^2}{\partial x^2} & \dfrac{\partial\phi^2}{\partial x^3} \end{vmatrix}_{\vec{x^*}} \neq 0, \tag{32}$$

which means x^* is a first-order degenerate point of the ϕ-mapping theory. (The case that x^* is a second-order degenerate point will be given in detail in the following section.) From $\phi^1 = 0$ and $\phi^2 = 0$, the implicit function theorem implies that there exists one and only one system of function relationships

$$x^2 = x^2(t, x^1), \qquad x^3 = x^3(t, x^1). \tag{33}$$

Substituting (33) into ϕ^1 and ϕ^2, we can obtain

$$\phi^b(t, x^1, x^2(t, x^1), x^3(t, x^1)) \equiv 0, \qquad b = 1, 2 \tag{34}$$

which give

$$\sum_{j=2}^{3}\phi^b_j x^j_0 = -\phi^b_0, \qquad \sum_{j=2}^{3}\phi^b_j x^j_1 = -\phi^b_1, \tag{35}$$

$$\sum_{j=2}^{3}\phi^b_j x^j_{00} = -\sum_{j=2}^{3}\left[2\phi^b_{j0}x^j_1 + \sum_{k=2}^{3}\left(\phi^b_{jk}x^k_0\right)x^j_1\right] - \phi^b_{01}, \tag{36}$$

$$\sum_{j=2}^{3}\phi^b_j x^j_{01} = -\sum_{j=2}^{3}\left[\phi^b_{j0}x^j_1 + \phi^b_{j1}x^j_0 + \sum_{k=2}^{3}\left(\phi^b_{jk}x^k_0\right)x^j_0\right] - \phi^b_{00}, \tag{37}$$

$$\sum_{j=2}^{3}\phi^b_j x^j_{11} = -\sum_{j=2}^{3}\left[2\phi^b_{j1}x^j_1 + \sum_{k=2}^{3}\left(\phi^b_{jk}x^k_1\right)x^j_1\right] - \phi^b_{11}, \tag{38}$$

where $b = 1, 2$; $j, k = 2, 3$; and

$$x^j_0 = \frac{\partial x^j}{\partial t}, \qquad x^j_1 = \frac{\partial x^j}{\partial x^1}, \qquad x^j_{00} = \frac{\partial^2 x^j}{\partial t^2}, \qquad x^j_{01} = \frac{\partial^2 x^j}{\partial t\partial x^1}, \qquad x^j_{11} = \frac{\partial^2 x^j}{(\partial x^1)^2}, \tag{39}$$

$$\phi^b_0 = \frac{\partial\phi^b}{\partial t}, \qquad \phi^b_1 = \frac{\partial\phi^b}{\partial x^1}, \qquad \phi^b_j = \frac{\partial\phi^b}{\partial x^j}, \qquad \phi^b_{00} = \frac{\partial^2\phi^b}{\partial t^2}, \qquad \phi^b_{01} = \frac{\partial^2\phi^b}{\partial t\partial x^1}, \tag{40}$$

$$\phi^b_{11} = \frac{\partial^2\phi^b}{(\partial x^1)^2}, \qquad \phi^b_{j0} = \frac{\partial^2\phi^b}{\partial t\partial x^j}, \qquad \phi^b_{j1} = \frac{\partial^2\phi^b}{\partial x^1\partial x^j}, \qquad \phi^b_{jk} = \frac{\partial^2\phi^b}{\partial x^j\partial x^k}. \tag{41}$$

From these expressions we can calculate the values of the first- and second-order partial derivatives of (33) with respect to t and x^1 at the bifurcation point $\vec{x^*}$.

Here we must note that the above discussions do not relate to the last component $\phi^3(\vec{x},t)$ of the vector order parameter $\vec{\phi}$. With the aim of finding the different directions of all branch curves at the bifurcation point, let us investigate the Taylor expansion of

$$F(t,x^1) = \phi^3(t,x^1,x^2(t,x^1),x^3(t,x^1)), \tag{42}$$

in the bifurcation point, which must vanish at the bifurcation point, i.e.,

$$F(t^*,x^{1*}) = 0. \tag{43}$$

From (42), the first-order partial derivatives of $F(t,x^1)$ is

$$\frac{\partial F}{\partial t} = \frac{\partial \phi^3}{\partial t} + \sum_{j=2}^{3} \frac{\partial \phi^3}{\partial x^j} x_0^j, \qquad \frac{\partial F}{\partial x^1} = \frac{\partial \phi^3}{\partial x^1} + \sum_{j=2}^{3} \frac{\partial \phi^3}{\partial x^j} x_1^j. \tag{44}$$

On the other hand, making use of (32), (35), (44) and Cramer's rule, it is not difficult to prove that the two restrictive conditions in (29) can be rewritten as

$$D\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = \frac{\partial F}{\partial x^1} D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = 0, \tag{45}$$

$$D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = \frac{\partial F}{\partial t} D^1\left(\frac{\phi}{x}\right)\bigg|_{(t^*,\vec{x}^*)} = 0. \tag{46}$$

By considering (32), the above equations give

$$\frac{\partial F}{\partial t}\bigg|_{(t^*,\vec{x}^*)} = 0, \qquad \frac{\partial F}{\partial x^1}\bigg|_{(t^*,\vec{x}^*)} = 0. \tag{47}$$

The second-order partial derivatives of the function $F(t,x^1)$ are easily found to be

$$\frac{\partial^2 F}{\partial t^2} = \phi_{00}^3 + \sum_{j=2}^{3}\left[2\phi_{j0}^3 x_0^j + \phi_j^3 x_{00}^j + \sum_{k=2}^{3}(\phi_{jk}^3 x_0^k) x_0^j\right], \tag{48}$$

$$\frac{\partial^2 F}{\partial t \partial x^1} = \phi_{11}^3 + \sum_{j=2}^{3}\left[\phi_{j0}^3 x_1^j + \phi_{j1}^3 x_0^j + \phi_j^3 x_{01}^j + \sum_{k=2}^{3}(\phi_{jk}^3 x_0^k) x_0^j\right], \tag{49}$$

$$\frac{\partial^2 F}{(\partial x^1)^2} = \phi_{11}^3 + \sum_{j=2}^{3}\left[2\phi_{j1}^3 x_1^j + \phi_j^3 x_{11}^j + \sum_{k=2}^{3}(\phi_{jk}^3 x_1^k) x_1^j\right], \tag{50}$$

which at $x^* = (t^*,\vec{x}^*)$ are denoted by

$$A = \frac{\partial^2 F}{\partial t^2}\bigg|_{(t^*,\vec{x}^*)}, \qquad B = \frac{\partial^2 F}{\partial t \partial x^1}\bigg|_{(t^*,\vec{x}^*)}, \qquad C = \frac{\partial^2 F}{(\partial x^1)^2}\bigg|_{(t^*,\vec{x}^*)}, \tag{51}$$

where $j,k = 2,3$ and

$$\phi_j^3 = \frac{\partial \phi^3}{\partial x^j}, \qquad \phi_{00}^3 = \frac{\partial^2 \phi^3}{\partial t^2}, \qquad \phi_{01}^3 = \frac{\partial^2 \phi^3}{\partial t \partial x^1}, \qquad \phi_{11}^3 = \frac{\partial^2 \phi^3}{(\partial x^1)^2}, \tag{52}$$

$$\phi_{j0}^3 = \frac{\partial^2 \phi^3}{\partial t \partial x^j}, \qquad \phi_{j1}^3 = \frac{\partial^2 \phi^3}{\partial x^1 \partial x^j}, \qquad \phi_{jk}^3 = \frac{\partial^2 \phi^3}{\partial x^j \partial x^k}. \tag{53}$$

According to the *Duan*'s topological current theory, the Taylor expansion of the solution of ϕ^3 in the neighborhood of the bifurcation point can generally be denoted as

$$A\left(x^1 - z_l^1\right)^2 + 2B\left(x^2 - z_l^2\right)(t - t^*) + (t - t^*)^2 = 0, \tag{54}$$

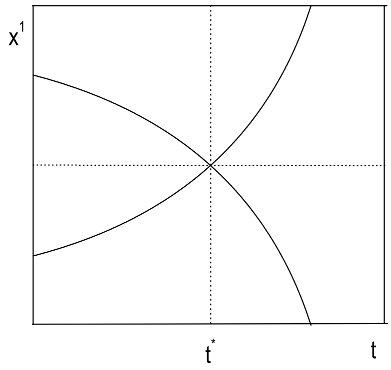


Figure 2. Projecting the world lines of magnetic monopoles onto the $(x^1 - t)$ plane. Two magnetic monopoles meet and then depart at the bifurcation point.

which is followed by

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + C = 0, \tag{55}$$

and

$$C\left(\frac{\mathrm{d}t}{\mathrm{d}x^1}\right)^2 + 2B\frac{\mathrm{d}t}{\mathrm{d}x^1} + A = 0, \tag{56}$$

where A, B and C are three constants. The solutions of equation (55) or equation (56) give different directions of the branch curves (world lines of the magnetic monopoles) at the bifurcation point. There are four kinds of important cases, which will show the physical meanings of the bifurcation points.

Case 1($A \neq 0$). For $\Delta = 4(B^2 - AC) > 0$, we get two different directions of the velocity field of magnetic monopoles

$$\left.\frac{\mathrm{d}x^1}{\mathrm{d}t}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \tag{57}$$

which are shown in figure 2. It is the intersection of two magnetic monopoles, which means that two magnetic monopoles meet and then depart from each other at the bifurcation point.

Case 2($A \neq 0$). For $\Delta = 4(B^2 - AC) = 0$, the direction of the velocity field of the magnetic monopole is only one

$$\left.\frac{\mathrm{d}x^1}{\mathrm{d}t}\right|_{1,2} = \frac{-B}{A}, \tag{58}$$

which includes three important situations. (a) One world line resolves into two world lines, i.e., one magnetic monopole splits into two magnetic monopoles at the bifurcation point (see figure 3(*a*)). (b) Two world lines merge into one magnetic monopole, i.e., two magnetic monopoles merge into one magnetic monopole at the bifurcation point (see figure 3(*b*)). (c) Two world lines tangentially contact, i.e., two magnetic monopoles tangentially encounter at the bifurcation point (see figure 3(*c*)).

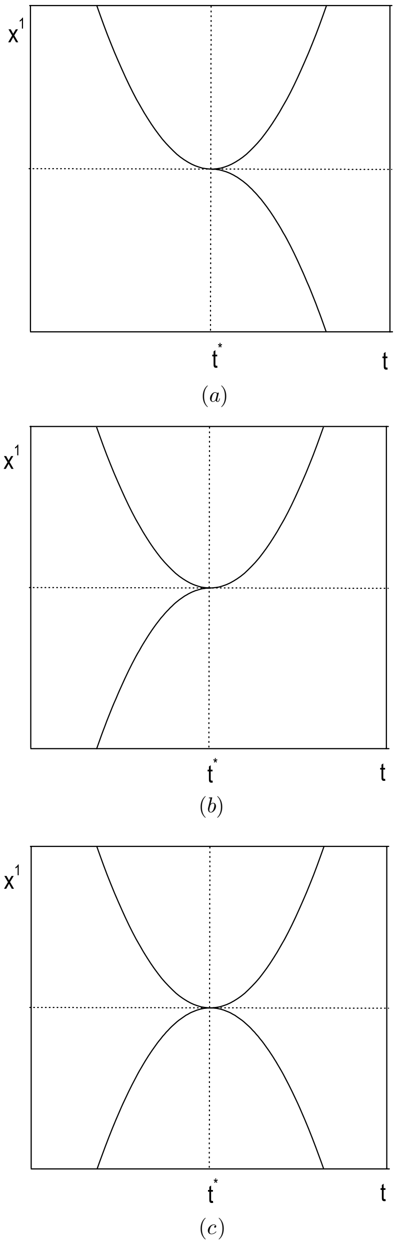


Figure 3. (*a*) One magnetic monopole splits into two magnetic monopoles at the bifurcation point. (*b*) Two magnetic monopoles merge into one magnetic monopole at the bifurcation point. (*c*) Two world line of magnetic monopoles tangentially intersect, i.e., two magnetic monopoles tangentially encounter at the bifurcation point.

Case 3$(A = 0, C \neq 0)$. For $\Delta = 4(B^2 - AC) = 0$, we have

$$\left.\frac{\mathrm{d}t}{\mathrm{d}x^1}\right|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, \qquad -\frac{2B}{C}. \tag{59}$$

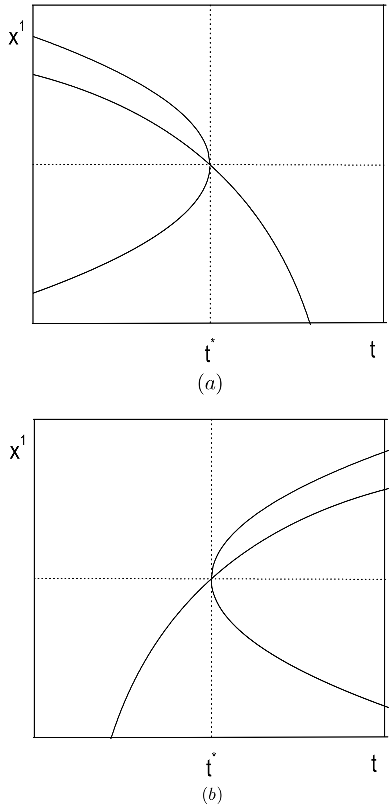


Figure 4. Two important cases of equation (59). (*a*) Three magnetic monopoles merge into one at the bifurcation point. (*b*) One magnetic monopole splits into three magnetic monopoles at the bifurcation point.

There are two important cases: (a) three world lines merge into one world line, i.e., three magnetic monopoles merge into a magnetic monopole at the bifurcation point (see figure 4(*a*)). (b) One world line resolves into three world lines, i.e., a magnetic monopole splits into three magnetic monopoles at the bifurcation point (see figure 4(*b*)).

Case 4$(A = C = 0)$. Equations (55) and (56) give, respectively

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}t}{\mathrm{d}x^1} = 0. \tag{60}$$

This case is obvious (see figure 5), and similar to case 3.

The above solutions reveal the evolution of the magnetic monopoles. Besides the encountering of the magnetic monopoles, i.e., two magnetic monopoles encounter and then depart at the bifurcation point along different branch curves (see figure 2 and figure 3(*c*)), it also includes splitting and merging of magnetic monopoles. When a multi-charged magnetic monopole moves through the bifurcation point, it may split into several magnetic monopoles along different branch curves (see figures 3(*a*), 4(*b*) and 5(*b*)). In contrast, magnetic monopoles can merge into a magnetic monopole at the bifurcation point (see figures 3(*b*) and 4(*a*)).

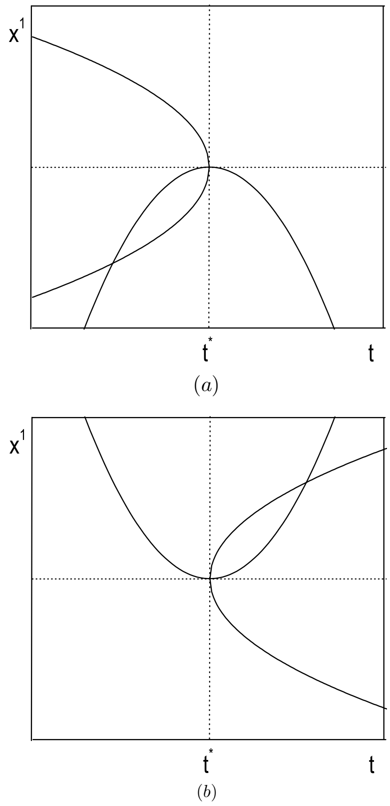


Figure 5. Two world lines intersect normally at the bifurcation point. This case is similar to figure 4. (*a*) Three magnetic monopoles merge into one at the bifurcation point. (*b*) One magnetic monopole splits into three magnetic monopoles at the bifurcation point.

At the same time, the remaining component can be deduced by

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = x_0^j + x_1^j \frac{\mathrm{d}x^1}{\mathrm{d}t}, \qquad j = 2, 3. \tag{61}$$

As in the previous work, the identical conversation of the topological charge shows the sum of the topological charge of these split magnetic monopoles must be equal to that of the original magnetic monopoles at the bifurcation point, i.e.,

$$\sum_i \beta_{l_i} \eta_{l_i} = \sum_f \beta_{l_f} \eta_{l_f}, \tag{62}$$

for fixed l. Furthermore, from the above studies, we see that the generation, annihilation and bifurcation of magnetic monopoles are not gradually changed, but suddenly changed at the critical points.

5. The bifurcation of a magnetic monopole at a second-order degenerate point

In the preceding section we studied the bifurcation of a magnetic monopole at a first-order degenerate point. In this section, we investigate the branching process of the magnetic charge current at a second-order degenerate point $x^* = (t^*, \vec{x}^*)$, at which the rank of the Jacobian matrix $\left[\frac{\partial \phi}{\partial x}\right]$ is

$$\operatorname{rank}\left[\frac{\partial \phi}{\partial x}\right]\Big|_{(t^*,\vec{x}^*)} = 3 - 2 = 1. \tag{63}$$

Suppose that

$$\frac{\partial \phi^1}{\partial x^3}\Big|_{(t^*,\vec{x}^*)} \neq 0. \tag{64}$$

With the same reasons as in obtaining (33), in the neighborhood of x^*, from $\phi^1(x) = 0$ we have the function relationship

$$x^3 = x^3(t, x^1, x^2). \tag{65}$$

In order to determine the values of the first- and second-order partial derivatives of x^3 with respect to t, x^1 and x^2, one can substitute the relationship (65) into $\phi^2(x) = 0$ and $\phi^3(x) = 0$. Then, we get

$$\begin{aligned} F_1(t, x^1, x^2) &= \phi^2(t, x^1, x^2, x^3(t, x^1, x^2)) = 0, \\ F_2(t, x^1, x^2) &= \phi^3(t, x^1, x^2, x^3(t, x^1, x^2)) = 0. \end{aligned} \tag{66}$$

For calculating the partial derivatives of the function F_1 and F_2 with respect to t, x^1 and x^2, one can take note of (65) and use six similar expressions to (47), i.e.,

$$\frac{\partial F_c}{\partial t}\Big|_{(t^*,\vec{x}^*)} = 0, \qquad \frac{\partial F_c}{\partial x^1}\Big|_{(t^*,\vec{x}^*)} = \phi, \qquad \frac{\partial F_c}{\partial x^2}\Big|_{(t^*,\vec{x}^*)} = \phi, \qquad c = 1, 2. \tag{67}$$

So the Taylor expansions of $F_1(t, x^1, x^2)$ and $F_2(t, x^1, x^2)$ can be written in the neighborhood of $(t^*, \vec{x}^*)$ by

$$\begin{aligned} F_c(t, x^1, x^2) \approx{}& A_{c1}(t - t^*)^2 + A_{c2}(t - t^*)(x^1 - x^{1*}) + A_{c3}(t - t^*)(x^2 - x^{2*}) \\ &+ A_{c4}(x^1 - x^{1*})^2 + A_{c5}(x^1 - x^{1*})(x^2 - x^{2*}) + A_{c6}(x^2 - x^{2*})^2 = 0, \end{aligned} \tag{68}$$

where $c = 1, 2$ and

$$\begin{aligned} &A_{c1} = \frac{1}{2}\frac{\partial^2 F_c}{\partial t^2}\Big|_{(t^*,\vec{x}^*)}, \qquad A_{c2} = \frac{\partial^2 F_c}{\partial t\partial x^1}\Big|_{(t^*,\vec{x}^*)}, \qquad A_{c3} = \frac{\partial^2 F_c}{\partial t\partial x^2}\Big|_{(t^*,\vec{x}^*)}, \\ &A_{c4} = \frac{1}{2}\frac{\partial^2 F_c}{(\partial x^1)^2}\Big|_{(t^*,\vec{x}^*)}, \qquad A_{c5} = \frac{\partial^2 F_c}{\partial x^1\partial x^2}\Big|_{(t^*,\vec{x}^*)}, \qquad A_{c6} = \frac{1}{2}\frac{\partial^2 F_c}{(\partial x^2)^2}\Big|_{(t^*,\vec{x}^*)}. \end{aligned} \tag{69}$$

Dividing (68) by $(t - t^*)^2$ and taking the limit $t \to t^*$, one obtains the two quadratic equations of $\frac{\mathrm{d}x^1}{\mathrm{d}t}$ and $\frac{\mathrm{d}x^2}{\mathrm{d}t}$,

$$A_{c1} + A_{c2}\frac{\mathrm{d}x^1}{\mathrm{d}t} + A_{c3}\frac{\mathrm{d}x^2}{\mathrm{d}t} + A_{c4}\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + A_{c5}\frac{\mathrm{d}x^1}{\mathrm{d}t}\frac{\mathrm{d}x^2}{\mathrm{d}t} + A_{c6}\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2 = 0, \tag{70}$$

and further, eliminating the variable $\frac{\mathrm{d}x^1}{\mathrm{d}t}$, one has the equation of $\frac{\mathrm{d}x^2}{\mathrm{d}t}$ in the form of a determinant

$$\begin{vmatrix} A_{14} & A_{15}v + A_{12} & A_{16}v^2 + A_{13}v + A_{11} & 0 \\ 0 & A_{14} & A_{15}v + A_{12} & A_{16}v^2 + A_{13}v + A_{11} \\ A_{24} & A_{25}v + A_{22} & A_{26}v^2 + A_{23}v + A_{21} & 0 \\ 0 & A_{14} & A_{25}v + a_{22} & A_{26}v^2 + A_{23}v + A_{21} \end{vmatrix} = 0, \tag{71}$$

with the variable $v = \frac{\mathrm{d}x^2}{\mathrm{d}t}$, which is a four-order equation of $\frac{\mathrm{d}x^2}{\mathrm{d}t}$

$$a_1\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^4 + a_2\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^3 + a_3\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right)^2 + a_4\left(\frac{\mathrm{d}x^2}{\mathrm{d}t}\right) + a_5 = 0. \tag{72}$$

Hence, different directions of the branch curves at the second-order degenerate point x^* is structured. The largest number of different branch curves is four, which means an original magnetic monopole with the topological quantum $\beta\eta$ can split into at most four particles at one time with charges $\beta_l\eta_l (l = 1, 2, 3, 4)$ satisfying

$$\beta_1\eta_1 + \beta_2\eta_2 + \beta_3\eta_3 + \beta_4\eta_4 = \beta\eta. \tag{73}$$

6. Conclusions

Our conclusions can be summarized as follows: first, in a charged two-component Bose–Einstein system, we obtained the dynamic form of a magnetic monopole and quantized the magnetic charge at the topological level in units of $\frac{\hbar c}{e}$. The topological quantum numbers are determined by the Hopf indices and Brouwer degrees (i.e. the winding numbers), which are topological numbers. Second, the evolution of magnetic monopoles is studied from the topological properties of a three-dimensional vector field $\vec{\phi}$. We find that there exist crucial cases of branch processes in the evolution of the magnetic monopoles when $D(\frac{\phi}{x}) \neq 0$, i.e., η_l is indefinite. This means that the magnetic monopoles generate or annihilate at the limit points and encounter, split or merge at the bifurcation points of the three-dimensional vector field $\vec{\phi}$, which shows that the magnetic monopoles system is unstable at these branch points. Third, we show the result that the velocity of a magnetic monopole is infinite when it is annihilating or generating, which is obtained only from the topological properties of the three-dimensional vector field $\vec{\phi}$. Fourth, we must point out that there exist two restrictions of the evolution of magnetic monopoles. One restriction is the conservation of the topological charge of the magnetic monopoles during the branch process (see equations (28) and (62)), the other is that the number of different directions of the world lines of magnetic monopoles is at most four at the bifurcation points (see equations (55) and (56)). The first restriction is already known, but the second is pointed out here for the first time to our knowledge. We hope that it can be verified in the future. Finally, we would like to point out that all the results in this paper have been obtained only from the viewpoint of topology without using any particular models or hypothesis.

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